

# Quantum History cannot be Copied

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## Abstract

We show that unitarity does not allow cloning of any two points in a ray. This has implication for cloning of the geometric phase information in a quantum state. In particular, the quantum history which is encoded in the geometric phase during cyclic evolution of a quantum system cannot be copied. We also prove that the generalized geometric phase information cannot be copied by a unitary operation. We argue that our result also holds in the consistent history formulation of quantum mechanics.

In quantum theory state of a single quantum is represented by not just a vector  $|\psi\rangle$  in a separable Hilbert space  $\mathcal{H}$ , but by a ray in the ray space  $\mathcal{R}$ . A ray is a set of equivalence classes of states that differ from each other by complex numbers of unit modulus. Thus the ray space  $\mathcal{R}$  is defined as  $\mathcal{R} = \{|\psi'\rangle : |\psi\rangle \sim |\psi'\rangle = c|\psi\rangle\}$ , where  $c \in \mathbf{C}$  is a group of non-zero complex numbers and  $|c| = 1$ . Given a quantum state  $|\psi\rangle$  we can generate a ray by the application of the ‘ray operator’  $R(c) = \exp[i\text{Arg}(c)|\psi\rangle\langle\psi|] = I + (c - 1)|\psi\rangle\langle\psi|$  such that  $R(c)|\psi\rangle = |\psi'\rangle$ . Geometrically, we can represent all these equivalent classes of states as points in a ray and all of them represent the same physical state. The set of rays of the Hilbert space  $\mathcal{H}$  is called the projective Hilbert space  $\mathcal{P} = \mathcal{H}/U(1)$ . If we have a continuous unitary time evolution of a quantum system  $|\psi\rangle \rightarrow U(t)|\psi\rangle$  then the evolution can be represented as an open curve  $\Gamma : t \rightarrow |\psi(t)\rangle$  in  $\mathcal{R}$  whose projection in  $\mathcal{P}$  is also an open curve  $\hat{\Gamma}$ . The quantum state at different times can belong to different rays. If the quantum state at two different times belongs to the same ray, then it may trace an open curve  $C$  in  $\mathcal{R}$ , but its projection in  $\mathcal{P}$  is a closed curve  $\hat{C}$ . Such an evolution is called a cyclic evolution.

In quantum information theory we view a quantum state  $|\psi\rangle$  as the carrier of both classical and quantum information. The fundamental unit of classical information is a bit and that of quantum information is a qubit. Classical bit can be copied but quantum bit or qubit cannot be copied. It is the linearity of quantum theory that does not allow us to produce a copy of an arbitrary quantum state [1,2]. Using unitarity one can prove that two non-orthogonal states cannot be copied either [3]. However, orthogonal quantum states like

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$|0\rangle$  and  $|1\rangle$  can be copied unitarily. Also, if we know a quantum state, we can make as many copies as we wish. The subject of quantum cloning is an active area of research. Recent studies have shown that one could make approximate copies by deterministic transformations [4] and exact copies by probabilistic operations [5,6]. Another important limitation on quantum information is that it is impossible to delete an unknown quantum state [7]. We know that classical bit or quantum bit in orthogonal states can be deleted against a copy. However, linearity of quantum theory prohibits deletion of a qubit against a copy.

Here, we ask the following question: If we can make a copy of  $|\psi\rangle$  can we make copy of an equivalent state  $|\psi'\rangle$  by the same machine? That is whether  $|\psi\rangle \otimes |\Sigma\rangle \rightarrow |\psi\rangle \otimes |\psi\rangle$  and  $|\psi'\rangle \otimes |\Sigma\rangle \rightarrow |\psi'\rangle \otimes |\psi'\rangle$  is possible by a single unitary operator, where  $|\Sigma\rangle$  is the blank state and  $|\psi'\rangle = c|\psi\rangle$ . Intuitively, one would say that since  $|\psi\rangle$  and  $c|\psi\rangle$  represent the same physical state from informational point of view it should be possible to make copies of  $|\psi\rangle$  and  $c|\psi\rangle$  by the same cloning machine. But surprisingly, this intuition is not correct. The proof is simple but nevertheless it is important. This has important implication in cloning of relative phase information. For example, we prove that the geometric phase information during cyclic evolution of a quantum system cannot be copied by unitary machine or any physical operation (a completely positive trace preserving map). Furthermore, we prove that the non-cyclic geometric phase cannot be copied during an arbitrary evolution of a quantum system. The important implication of our theorems is that history of a quantum system which is encoded in the geometric phase cannot be copied. Interestingly, we argue that our result also holds in the consistent histories formulation of quantum theory where the geometric phase appears naturally in the histories.

**Theorem 1:** *In general, two points in a ray cannot be cloned by a single unitary machine.*

*Proof:* A ray is an equivalence classes of states up to global phases. Thus, rather than the usual cloning machine having the form  $|\psi\rangle \otimes |\Sigma\rangle \rightarrow |\psi\rangle \otimes |\psi\rangle$  we should really allow for

$$|\psi\rangle \otimes |\Sigma\rangle \rightarrow e^{i\theta(\psi)} |\psi\rangle \otimes |\psi\rangle, \quad (1)$$

where  $\theta(\psi)$  can be an arbitrary function of  $|\psi\rangle$ . Now, for two equivalent states  $|\psi\rangle$  and  $|\psi'\rangle$  if we take the transformation

$$\begin{aligned} |\psi\rangle \otimes |\Sigma\rangle &\rightarrow |\psi\rangle \otimes |\psi\rangle \\ |\psi'\rangle \otimes |\Sigma\rangle &\rightarrow |\psi'\rangle \otimes |\psi'\rangle, \end{aligned} \quad (2)$$

then this says that which specific member of the equivalence class we are in that needs to be preserved. One might argue that this would be against the spirit of having an equivalence class in the first place where any member of the equivalence class may be substituted with any other at any time. So, the general cloning transformation may be given by (1). Note that this ambiguity does not arise in usual cloning literature that describe approximate clones because there one asks only about the reduced state of each subsystem. It should be mentioned that we can also use the transformation (2) to prove our theorem but (1) is more general way of stating the action of cloning map on a ket. As we know for a fixed state vector, an overall phase is not important, since physical states are density matrices and not vectors. Indeed the cloning maps given by (1) and (2) are equivalent. Unless there is confusion, in this paper, we use sometime transformations (1) or (2).

Let us consider two equivalent classes of states  $|\psi\rangle$  and  $|\psi'\rangle$ . The cloning transformation for two points in a ray is now given by

$$\begin{aligned} |\psi\rangle \otimes |\Sigma\rangle &\rightarrow e^{i\theta(\psi)} |\psi\rangle \otimes |\psi\rangle \\ |\psi'\rangle \otimes |\Sigma\rangle &\rightarrow e^{i\theta(\psi')} |\psi\rangle \otimes |\psi\rangle \end{aligned} \quad (3)$$

Since unitarity preserves the inner product we must have  $c = \exp[i\theta(\psi) - \theta(\psi')]$ . However, this cannot hold for arbitrary values of  $c, \theta(\psi)$  and  $\theta(\psi')$ . This proves that two equivalent states cannot be copied by the same machine even if we know the state. The physical meaning of this theorem is that the relative phase between two points in a ray cannot be copied by a unitary machine.

The ‘no-cloning theorem for a ray’ will have implication for cloning of relative phase information in quantum systems. In recent years, there have been considerable interest in the study of the relative phases and in particular the Berry phases [8] in quantum systems. It is also hoped that the Berry phase which is of geometric origin may be used in design of robust logic gates in quantum computation. The original Berry phase was discovered in the context of quantum adiabatic theorem [10,11]. It is basically an extra phase that the system acquires when the Hamiltonian is slowly changed cyclically over one time period. The Berry phase is independent of the detailed dynamics of the system and is of purely geometric in origin. This arises due to the non-trivial curvature of the parameter space in which the state vector is transported around a closed loop. An early discovery of the geometric phase was made by Pancharatnam in the context of interference of light [9]. The Berry phase was then generalized to non-adiabatic but cyclic evolutions of quantum system by Aharonov and Anandan [12]. In fact, now we know that the geometric phase appears in much more general context than it was thought before [13–15].

Consider the unitary time evolution of a quantum system where the state vector evolves as  $|\psi(0)\rangle \rightarrow |\psi(t)\rangle = U(t)|\psi(0)\rangle$  such that at  $t = T$ ,  $|\psi(T)\rangle = e^{i\Phi}|\psi(0)\rangle$ ,  $\Phi$  being the total phase. That is to say that the quantum system at  $t = T$  comes back to its original state apart from a phase factor. As we know such an evolution is called the cyclic evolution. Even though  $|\psi(0)\rangle$  and  $|\psi(T)\rangle$  are equivalent it is the relative phase  $\Phi$  between them that is observable. The total phase  $\Phi$  that the system acquires during a cyclic evolution is composed of two phases, one is the dynamical phase  $\delta$  and the other is the geometric phase  $\beta(\hat{C})$ . This  $\beta(\hat{C})$  is also known as the Aharonov-Anandan (AA) phase [12]. Thus the total phase is given by

$$\Phi = \delta + \beta(\hat{C}), \quad (4)$$

where  $\delta$  is the dynamical phase and  $\beta(\hat{C})$  is the geometric phase. The dynamical phase is given by

$$\delta = -\frac{1}{\hbar} \int_0^T \langle \psi(t) | H | \psi(t) \rangle dt. \quad (5)$$

It represents, in a sense, an ‘internal clock’ of the quantum system. On the other hand the geometric phase is given by

$$\beta(\hat{C}) = i \oint \langle \tilde{\psi}(t) | \dot{\tilde{\psi}}(t) \rangle dt = i \oint \langle \tilde{\psi} | d\tilde{\psi} \rangle, \quad (6)$$

where  $|\tilde{\psi}(t)\rangle = \exp(-if(t))|\psi(t)\rangle$  with  $f(t)$  being any smooth function that satisfies  $f(T) - f(0) = \Phi$ . Here,  $i\langle\tilde{\psi}|d\tilde{\psi}\rangle$  is the differential connection-form that gives rise to geometric phase. It is gauge invariant, reparameterization invariant, and depends only on the closed curve  $\hat{C}$  in the projective Hilbert space  $\mathcal{P}$  of the quantum system. Unlike the dynamical phase, the geometric phase indeed depends on the path of the evolution [17] and is a non-integrable quantity.

In this context, if we are able to clone  $|\psi(0)\rangle$ , one may think that we can also clone  $|\psi(T)\rangle$  as they really belong to the same ray. This is because during a cyclic evolution the system starts from a ray and after a time period  $T$  comes back to the same ray but at a different point. Now, application of our theorem tells us that we cannot clone  $|\psi(0)\rangle$  and  $|\psi(T)\rangle$  by a single unitary machine.

**Proposition:** *Quantum history which is encoded in the geometric phase during cyclic evolution of a quantum system cannot be copied by a unitary transformation.*

*Proof:* Suppose we could copy  $|\psi(0)\rangle$  and  $|\psi(T)\rangle$  by a unitary machine. That is

$$\begin{aligned} |\psi(0)\rangle \otimes |\Sigma\rangle &\rightarrow |\psi(0)\rangle \otimes |\psi(0)\rangle \\ |\psi(T)\rangle \otimes |\Sigma\rangle &\rightarrow |\psi(T)\rangle \otimes |\psi(T)\rangle. \end{aligned} \quad (7)$$

By unitarity, Eq(7) implies that we must have  $\langle\psi(0)|\psi(T)\rangle \rightarrow \langle\psi(0)|\psi(T)\rangle^2$ , i.e.,  $\exp(i\Phi) \rightarrow \exp(2i\Phi)$ . However, this is not possible by a unitary machine. If it is possible, then that would mean  $\exp(i\Phi) = 1$  which is not true in general. Moreover, if the system undergoes parallel transportation then it acquires a pure geometric phase and we will have  $\Phi = \beta(\hat{C})$  [12]. Here we would like to mention that an *arbitrary quantum state cannot be parallel transported* either. (This is another no-go theorem. See the notes in the end). Then the no-cloning theorem for a ray tells us that we cannot clone the geometric phase information of a quantum system during a cyclic evolution. Since the geometric phase attributes memory to a quantum system it remembers the history of the evolution. This then implies that the *quantum history cannot be copied*. Therefore, even if we can make copy of a known quantum states we cannot copy its history. The only way to copy the history is to first make a copy of the state and then pass the copied quantum system through the same cycle again. Physical reason for this impossibility is the following. We are able to make copy of  $|\psi(0)\rangle$  because we have complete knowledge of it. But the geometric phase not only depends on  $|\psi(0)\rangle$  but also on the path of the evolution that the system has undergone in the past. Unless we have knowledge of the past history we cannot copy the geometric phase.

In fact, we can prove that quantum history cannot be copied by any physical operation. A physical operation in quantum theory we mean a completely positive (CP) trace preserving mapping.

**Theorem 2:** *In general, geometric phase during a cyclic evolution cannot be copied by a completely positive map.*

*Proof:* We know that by including ancilla, any CP map can be realized as a unitary evolution in an enlarged Hilbert space. Consider the cloning of  $|\psi(0)\rangle$  and  $|\psi(T)\rangle$  including ancilla. This may be given by

$$\begin{aligned}
|\psi(0)\rangle \otimes |\Sigma\rangle \otimes |A\rangle &\rightarrow |\psi(0)\rangle \otimes |\psi(0)\rangle \otimes |A(0)\rangle \\
|\psi(T)\rangle \otimes |\Sigma\rangle \otimes |A\rangle &\rightarrow |\psi(T)\rangle \otimes |\psi(T)\rangle \otimes |A(T)\rangle,
\end{aligned} \tag{8}$$

where  $|A\rangle$  is the initial state and  $|A(0)\rangle$ ,  $|A(T)\rangle$  are the final states of the ancilla. By unitarity in the enlarged Hilbert space, we have

$$e^{i\Phi} = e^{2i\Phi} \langle A(0)|A(T)\rangle. \tag{9}$$

This cannot hold in general, hence it is impossible to copy quantum history by any physical operation. However, if it so happens that environment also undergoes a cyclic evolution and acquires equal and opposite relative phase as that of the quantum system, i.e.,  $|A(T)\rangle = \exp(-i\Phi)|A(0)\rangle$ , then possibly quantum history can be copied by a completely positive map.

Next, we prove that the geometric phase information during a general quantum evolution cannot be copied. When a quantum system evolves in time it traces an open path  $\Gamma : t \rightarrow |\psi(t)\rangle$  in the ray space  $\mathcal{R}$  and the quantum state at different times belong to different rays. The projection of the evolution path  $\Gamma$  in the projective Hilbert space  $\mathcal{P}$  is also an open path  $\hat{\Gamma}$ . For example, consider the time evolution  $|\psi(0)\rangle \rightarrow |\psi(t)\rangle$ . The evolution under consideration *need not be adiabatic, cyclic, and even unitary*. All that is required is that there is a linear map and evolution curve should be smooth with a inner product defined over the Hilbert space. In this case the initial and final states are not equivalent. For any two non-orthogonal states the relative phase (or total phase difference) between them is given by the Pancharatnam phase  $\Phi_P = \text{Arg}\langle\psi(0)|\psi(t)\rangle$  [9,13]. It is always possible to write the total phase  $\Phi_P$  as sum of the dynamical phase  $\Phi_D$  and the geometric phase  $\Phi_G$ . The dynamical phase is given by

$$[\Phi_D]_0^t = -i \int \langle\psi(t)|\dot{\psi}(t)\rangle dt. \tag{10}$$

It depends on the detailed dynamics that the system is undergoing. The geometric phase is given by

$$\begin{aligned}
[\Phi_G]_0^t &= \text{Arg}\langle\psi(0)|\psi(t)\rangle + i \int \langle\psi(t)|\dot{\psi}(t)\rangle dt \\
&= i \int \langle\chi(t)|\dot{\chi}(t)\rangle dt = i \int \langle\chi|d\chi\rangle,
\end{aligned} \tag{11}$$

where  $|\chi(t)\rangle = \frac{\langle\psi(t)|\psi(0)\rangle}{|\langle\psi(t)|\psi(0)\rangle|} |\psi(t)\rangle$  is a reference-section introduced in [15,16]. Here,  $i\langle\chi|d\chi\rangle$  is the differential connection-form that gives rise to the most general geometric phase. It is again  $U(1)$  gauge invariant, reparameterization invariant, and depends only on the geometry of the open curve  $\hat{\Gamma}$  in the projective Hilbert space  $\mathcal{P}$  of the quantum system. It can be shown that it is a *non-additive quantity* which in turn implies that the system remembers along which path it has been transported. Thus, the most general geometric phase remembers the history of quantum system and attributes a memory to the quantum system. Next we prove the following.

**Theorem 3:** *Quantum history which is encoded in the generalized geometric phase during arbitrary evolution of a quantum system cannot be copied unitarily.*

*Proof:* Consider a sequence of time evolution of a quantum system from  $t = 0$  to  $t_1$  and then from  $t_1$  to  $t_2$ . Thus, we have the time evolution  $|\psi(0)\rangle \rightarrow |\psi(t_1)\rangle \rightarrow |\psi(t_2)\rangle$ . Suppose, we want to clone the quantum states  $|\psi(0)\rangle$ ,  $|\psi(t_1)\rangle$  and  $|\psi(t_2)\rangle$  by a unitary machine. Then we will have the following cloning transformation

$$\begin{aligned} |\psi(0)\rangle \otimes |\Sigma\rangle &\rightarrow |\psi(0)\rangle \otimes |\psi(0)\rangle \\ |\psi(t_1)\rangle \otimes |\Sigma\rangle &\rightarrow |\psi(t_1)\rangle \otimes |\psi(t_1)\rangle \\ |\psi(t_2)\rangle \otimes |\Sigma\rangle &\rightarrow |\psi(t_2)\rangle \otimes |\psi(t_2)\rangle. \end{aligned} \quad (12)$$

Now, by unitarity, taking all the inner products we will have

$$\langle\psi(0)|\psi(t_1)\rangle\langle\psi(t_1)|\psi(t_2)\rangle\langle\psi(t_2)|\psi(0)\rangle = [\langle\psi(0)|\psi(t_1)\rangle\langle\psi(t_1)|\psi(t_2)\rangle\langle\psi(t_2)|\psi(0)\rangle]^2. \quad (13)$$

Let us define a complex quantity called as the three point Bargmann invariant  $\Delta^{(3)}$  as

$$\Delta^{(3)} = \langle\psi(0)|\psi(t_1)\rangle\langle\psi(t_1)|\psi(t_2)\rangle\langle\psi(t_2)|\psi(0)\rangle. \quad (14)$$

The three point Bargmann invariant remains the same under unitary and antiunitary transformation and plays an important role in the kinematic approach to the theory of geometric phases developed by Mukunda and Simon [14]. Now taking the argument of both the sides of Eq(13), we will have  $\text{Arg}\Delta^{(3)} \rightarrow 2\text{Arg}\Delta^{(3)}$  which implies that  $\text{Arg}\Delta^{(3)} = 0$ . But  $\text{Arg}\Delta^{(3)}$  is nothing but the excess geometric phase that the system may acquire in going from  $t_0 = 0$  to  $t_1$  and then from  $t_1$  to  $t_2$  instead of going from  $t_0 = 0$  to  $t_2$  directly [15]. More precisely, let  $[\Phi_G]_0^{t_1}$  is the geometric phase that the system acquires during the evolution from time  $t_0 = 0$  to  $t_1$ ,  $[\Phi_G]_{t_1}^{t_2}$  is the geometric phase that the system acquires during the evolution from time  $t_1$  to  $t_2$ , and  $[\Phi_G]_0^{t_2}$  is the geometric phase that the system acquires during the evolution directly from time  $t_0 = 0$  to  $t_2$ . An important property of the geometric phase is that it is *non-additive*. We will indeed have  $[\Phi_G]_0^{t_1} + [\Phi_G]_{t_1}^{t_2} \neq [\Phi_G]_0^{t_2}$ . Thus, the excess geometric phase given by

$$[\Phi_G]_0^{t_1} + [\Phi_G]_{t_1}^{t_2} - [\Phi_G]_0^{t_2} = \text{Arg}[\langle\psi(0)|\psi(t_1)\rangle\langle\psi(t_1)|\psi(t_2)\rangle\langle\psi(t_2)|\psi(0)\rangle] = \text{Arg}\Delta^{(3)}. \quad (15)$$

In the above the dynamical phase has disappeared because that is simply an additive quantity. Thus,  $\text{Arg}\Delta^{(3)} \neq 0$ . Hence this shows that the quantum history encoded in the generalized geometric phase during an arbitrary quantum evolution cannot be copied.

Our result has implication in the context of consistent history formulation of quantum mechanics developed by Griffiths [18], Omne [19] and Gell-Mann and Hartle [20]. Recently, it has been shown by Anastopoulos and Savvidou that the geometric phase is manifested in the probabilistic structure of histories. Specifically, they have shown that the geometric phase is the basic building block of the interference phase between pair of histories [21].

In consistent history approach, typically a history can be thought of as a sequence of properties or events which correspond to a time-ordered sequence of propositions about the quantum system. These events are represented by projectors  $\Pi_1, \Pi_2, \dots, \Pi_n$  on the Hilbert space  $\mathcal{H}$  at a succession of times  $t_1 < t_2 < \dots < t_n$ . These projectors at different times need not commute. One defines the space of all histories by a history Hilbert space  $\mathcal{H}_h = \mathcal{H}_{t_i}^{\otimes t_i}$  which consists of tensor product of copies of the Hilbert space  $\mathcal{H}_{t_i}$  at  $i$ th instant of time. On the history Hilbert space one represents history as a projector  $P = \Pi_1 \otimes \Pi_2 \dots \Pi_n$ . The

meaning of such a history is that events  $\Pi_i$  occurs in the system at time  $t_i$ , respectively. One assigns a realistic interpretation to such a history provided certain consistency conditions are satisfied.

By introducing Heisenberg projector  $\Pi_i(t_i) = U(t_i)^\dagger \Pi_i U(t_i)$  one can define a weight operator  $C_P$  to each history defined by

$$C_P = U(t_n)^\dagger \Pi_n U(t_n) \cdots U(t_1)^\dagger \Pi_1 U(t_1) = \Pi_n(t_n) \cdots \Pi_1(t_1). \quad (16)$$

Given a pair of weight operators  $C_P$  and  $C_{P'}$  for two histories one defines an important quantity –that is the decoherence functional which is a complex valued function  $d(P, P')$  of history propositions  $P, P'$  that measures the quantum mechanical interference between them. More precisely, the decoherence functional is a mapping  $d : \mathcal{H}_h \times \mathcal{H}_h \rightarrow \mathbb{C}$  that satisfies the following conditions [22]:

1.  $d(P, P') = d(P', P)^*$  for all  $P, P'$ . This is hermiticity.
2.  $d(P, P) \geq 0$  for all  $P$ . This is positivity.
3. If  $P$  and  $P'$  are orthogonal, then for all  $P''$ ,  $d(P \oplus P', P'') = d(P, P'') + d(P', P'')$ . This is additivity.
4.  $d(I, I) = 1$ . This is normalization.

One way of defining the decoherence functional is

$$d(P, P') = \text{Tr}(C_P \rho_0 C_{P'}^\dagger) \quad (17)$$

for an initial density matrix  $\rho_0$ . This can be interpreted as a probability in standard quantum theory under certain conditions. When  $d(P, P') = 0$  for  $P \neq P'$  in a set of histories that satisfies  $\sum_i \Pi_i = 1$  and  $PP' = P\delta_{PP'}$ , then under this condition  $d(P, P)$  can be regarded as the probability that the history proposition  $P$  is true. The decoherence functional  $d(P, P')$  can be thought of as the degree of interference between the histories  $P$  and  $P'$ .

To see how the geometric phase appears in the consistent history formulation, consider the time ordered events with projectors at different times as  $\Pi_0, \Pi_1, \dots, \Pi_n$ . If one assumes that the projectors are fine grained, they can be represented as elements of the projective Hilbert space  $\mathcal{P}$ , i.e.,  $\Pi_i = \Pi_{t_i} = |\psi(t_i)\rangle\langle\psi(t_i)|$ . If one neglects the dynamical evolution, i.e., set the Hamiltonian equal to zero, then trace of the weight operator is given by

$$\text{Tr} C_P = \langle\psi(0)|\psi(t_n)\rangle\langle\psi(t_n)|\psi(t_{n-1})\rangle \cdots \langle\psi(t_1)|\psi(0)\rangle. \quad (18)$$

By assuming the number of time steps  $n$  very large and each time steps differs by  $\delta t$  with  $\delta t \sim O(1/n)$  one can approximate this by a continuous time history and we have

$$\text{Arg}[\text{Tr} C_P] = \Phi_G. \quad (19)$$

That is to each history one can assign a geometric phase [21]. Actually, one can understand the origin of the geometric phase in history formulation as follows. Note that

$$\text{Tr} C_P = \Delta^{(n+1)*}, \quad (20)$$

where  $\Delta^{(n+1)}$  is the  $(n+1)$ -point Bargmann invariant defined as

$$\Delta^{(n+1)} = \langle\psi(0)|\psi(t_1)\rangle\langle\psi(t_1)|\psi(t_2)\rangle \cdots \langle\psi(t_{n-1})|\psi(t_n)\rangle. \quad (21)$$

We already know that in the standard quantum theory the Bargmann invariant represents the excess geometric phase if system undergoes sequence of times evolution between  $t_0 < t_1 < \dots < t_n$  and a direct evolution from  $t_0$  to  $t_n$ . This is true even if we do not assume that the Hamiltonian is zero. So in the consistency history formulation the appearance of geometric phase is natural. It has been also shown that the information about the geometric phase for a set of histories is sufficient to reconstruct the decoherence functional [21]. Now, from our theorem we know that geometric phase cannot be copied, then this also holds for the decoherence functional in the consistent history formulation.

In conclusion, we have proved that two equivalence classes of states representing the same physical state cannot be cloned by a unitary operator. It is the relative phase between two points in a ray that is impossible to clone. Even though the proof is really simple its implications may be important. During cyclic evolution of a quantum system the initial and final states are equivalent and can be represented as two points in a ray. One implication of our theorem is that the geometric phase information during a cyclic evolution cannot be copied. Since the geometric phase attributes a memory to a quantum system and remembers the history, this suggests that even though a state can be copied its quantum history cannot be copied. We have also shown that the geometric phase information during a cyclic evolution cannot be copied by a physical operation. In addition, we have proved that the geometric phase during arbitrary quantum evolution cannot be copied by a unitary machine. Interestingly, we have argued that our result also holds in the consistent history formulation of quantum theory. We hope that the impossibility of copying quantum history may have some application in quantum cosmology.

*Notes:* Here we prove that in general an arbitrary quantum state cannot be parallel transported. The parallel transport condition for a pure quantum state is that it never rotates locally (so it does not acquire any phase infinitesimally) but can under go a net rotation globally. This reflects the curvature of the quantum state space, i.e., the projective Hilbert space  $\mathcal{P}$  in which the vector is parallel transported. Thus, during a parallel transportation a state can acquire a phase if brought back to its original position along a closed path  $\hat{C}$ . This phase is essentially the holonomy angle or the geometric phase  $\beta(\hat{C})$ . Mathematically, the parallel transport condition for a vector  $|\psi(t)\rangle$  can be expressed as  $\langle\psi(t)|\dot{\psi}(t)\rangle = 0$ . If  $|\psi(t)\rangle$  satisfies this then  $|\psi(T)\rangle = \exp(i\beta(\hat{C}))|\psi(0)\rangle$  [12] during a cyclic evolution and  $\langle\psi(0)|\psi(t)\rangle = |\langle\psi(0)|\psi(t)\rangle| \exp[\Phi_G]$  during arbitrary non-cyclic evolution [15,16].

Suppose we have a set of known orthonormal bases  $\{|\psi_n(t)\rangle\}$  with  $\langle\psi_n(t)|\psi_m(t)\rangle = \delta_{nm}$  for all  $t$ . If these bases undergo parallel transportation then they satisfy  $\langle\psi_n(t)|\dot{\psi}_n(t)\rangle = 0$ . Now let  $|\psi(t)\rangle$  be an arbitrary state:  $|\psi(t)\rangle = \sum_{n=1}^N c_n(t)|\psi_n(t)\rangle \in \mathcal{H}^N$ . The question is if  $\langle\psi_n(t)|\dot{\psi}_n(t)\rangle = 0$  does that mean  $\langle\psi(t)|\dot{\psi}(t)\rangle = 0$ ? The answer is no. To be explicit we have

$$\langle\psi(t)|\dot{\psi}(t)\rangle = \sum_n c_n^* \dot{c}_n + \sum_{nm, n \neq m} c_m^* c_n \langle\psi_m(t)|\dot{\psi}_n(t)\rangle \neq 0. \quad (22)$$

Thus, in general an arbitrary quantum state cannot be parallel transported. In other words, *universal parallel transportation machine cannot exist*. However, there can be some special cases where it can be. When  $c_n(t)$ 's are time-independent and  $\langle\psi_m(t)|\dot{\psi}_n(t)\rangle = 0$  for  $m \neq n$  then an arbitrary state can also undergo parallel transportation. This is a both necessary and sufficient condition. It would be an interesting problem by itself to find what kind of Hamiltonian would satisfy the parallel transport condition for an arbitrary quantum state.



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